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ON THE DECOMPOSITION NUMBERS OF THE FINITE GENERAL LINEAR GROUPS. II

BY

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ABSTRACT. Let q be a prime power, $G = \mathrm{GL}_n(q)$ and let r be a prime not dividing q. Using representations of Hecke algebras associated with symmetric groups over arbitrary fields, the r-modular irreducible G-modules are classified. The decomposition matrix D of G (with respect to r) is partly described in terms of decomposition matrices of Hecke algebras, and it is shown that D is lower unitriangular, provided the irreducible characters and irreducible Brauer characters of G are suitably ordered.

Introduction. It is a well-known theorem that the decomposition matrices of symmetric groups with respect to any prime p are always lower unitriangular.

Our main theorem in this paper says that the same is true for the general linear groups for all primes different from the describing characteristic, i.e. for all primes r not dividing q, where $G = \mathrm{GL}_n(q)$ for some $n \in \mathbf{N}$ and some prime power q.

The Hecke algebra $R[W]$ of a symmetric group W over an integral domain R plays the key role for the proof of the theorem. If r divides $q - 1$, then the occurring Hecke algebras $R[W]$ over a field R of characteristic r are isomorphic to the usual group algebra RW over R. Thus the representation theory of symmetric groups has been used in [**2**] to give a close connection between the decomposition matrix of W with respect to r and the decomposition matrix $\mathrm{GL}_n(q)$.

Meanwhile, representation theory of Hecke algebras $R[W]$ over arbitrary integral domains R has been developed so far in [**3**] that we are able to generalize [**2**] to arbitrary primes r not dividing q.

However the main work is done in [**2** and **3**]. So using [**3**] most proofs in [**2**] carry over with only slight modifications to the more general situation here. So we shall frequently refer to [**2**] and omit the proofs in this paper.

It should be remarked that G. James proved in [**9**], with entirely different methods, results on the decomposition modulo r of unipotent characters of G, indicating a strong connection with representation theory of Hecke algebras as well. G. James and the author will combine both methods in a forthcoming paper, which gives a better insight into the connections mentioned above and more details on the decomposition matrices of general linear groups.

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1. Preliminaries. Let $G = \mathrm{GL}_n(q)$, the general linear group of rank n over the field F with q elements, q a prime power. Throughout we will adopt the notation of [**2**]. In particular r will be a prime not dividing q, and $(\bar{R}, R, K)$ is an r-modular split system for G.

If s is a semisimple element of G, then s is conjugate to $\prod m_\Lambda(s)(\Lambda)$, where the product has to be taken over the set $\mathscr{F}$ of monic irreducible polynomials over F, (Λ) denotes the companion matrix of $\Lambda \in \mathscr{F}$, and $m_\Lambda(s)$ the multiplicity of Λ as an elementary divisor of s. As in [**2**] we fix a linear order on $\mathscr{F}$ and take all occurring products, sums, etc. over $\mathscr{F}$ in this order, if not otherwise stated. Recall that the centralizer $C_G(s)$ of s in G is isomorphic to $\prod \mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda})$. The Weyl group of $C_G(s)$ is denoted by W_s. As in [**2**] we consider all occurring Weyl groups as subgroups of G.

Define the Levi subgroup L_s of G setting $L_s = \prod \mathrm{GL}_{\deg \Lambda}(q)^{m_\Lambda(s)}$ and the parabolic subgroup P_s with Levi decomposition $P_s = L_s U_s$ by $P_s = L_s U_0$, where U_0 is the subgroup of all upper unitriangular matrices in G.

To s we construct as in [**2**] a certain irreducible character ψ of L_s using Deligne-Lusztig operators. We choose the RL_s-lattice S_0 affording ψ as in [**2**] and denote the pull back of S_0 to P_s by S, respectively by S_s. Note that ψ is a cuspidal character of L_s by [**2**, 1.3], i.e. the trivial character 1_U does not occur as constituent in the restriction of ψ to U for the unipotent radical U of any proper parabolic subgroup of L_s (compare [**11**, Chapter 4]).

Let b_s be the block of RP_s, which contains S. In [**2**] the block idempotent of b_s has been denoted by e_s. Since we need the letter e in a different meaning in the following, we change notation and denote the block idempotent of b_s by ε_s. We shall frequently refer to the following:

1.1 Hypothesis. (i) $\bar{R}S$ is an irreducible $\bar{R}P_s$-module.
(ii) Let $V \in b_s$ be an irreducible constitutent of $KS_{P_s}^G$. Then $V = KS$.

By [**7**, **2**] $\mathrm{End}_{\tilde{R}G}(\tilde{R}S^G) = \tilde{R}[W_s]$ for $\tilde{R} = K$, and, if 1.1 holds, for every choice of $\tilde{R} \in \{\bar{R}, R, K\}$, where $\tilde{R}[W_s]$ denotes the Hecke algebra of W_s with respect to $\tilde{R}$ as defined in [**2**, §3]. We recall that $K[W_s] = KW_s$, the usual group algebra of W_s over K. Since $W_s = \prod \sigma_{m_\Lambda(s)}$, where $\sigma_{m_\Lambda(s)}$ denotes the symmetric group acting on $m_\Lambda(s)$ letters, the irreducible $K[W_s]$-modules are indexed by multipartitions $\lambda = \prod \lambda_\Lambda$, where λ_Λ is a partition of $m_\Lambda(s)$ for $\Lambda \in \mathscr{F}$. The character of the corresponding irreducible constituent of KS^G is denoted by $\chi_{s,\lambda}$.

For a natural number m we write shortly $\mu \vdash m$ if μ is a partition of m. Let $M_s = \{\lambda = \prod \lambda_\Lambda | \lambda_\Lambda \vdash m_\Lambda(s)\}$. Then $\{\chi_{s,\lambda} | \lambda \in M_s\}$ is called the geometric conjugacy class of s and is denoted by $(s)^G$. If t and s are semisimple conjugate elements of G, then $(t)^G = (s)^G$. Moreover the set of characters $\chi_{s,\lambda}$, where s runs through a set of representatives of G-conjugacy classes of semisimple elements and λ through M_s, is a full set of nonisomorphic irreducible characters of G by [**5**] (compare [**4**]).

In general $\tilde{R}[W_s] \not\cong \tilde{R}W_s$ $(\tilde{R} \in \{\bar{R}, R\})$, but the representation theory of $\tilde{R}[W_s]$ is a remarkable q-analogue to the representation theory of symmetric groups. It has been developed in [6] for $\tilde{R} = K$ and in [3] for arbitrary domains $\tilde{R}$. We are going to give a brief survey on the main results proved in [3].

So let $\tilde{R} \in \{\bar{R}, R, K\}$. For each $\lambda \in M_s$ we may define a Specht module $S^\lambda = S^\lambda_{\tilde{R}}$ generated by a certain element $z_\lambda = z_{\lambda,\tilde{R}}$ of $\tilde{R}[W_s]$. Note that for $\tilde{R} = R$, the definitions of S^λ and z_λ in [2 and 3] coincide. S^λ has a standard basis (independent of $\tilde{R}$ and q) [3, 5.6]. In particular, $S^\lambda_{\bar{R}} = z_{\lambda,\bar{R}}\bar{R}[W_s]$ is a reduction $\bmod r$ of $S^\lambda_K = z_{\lambda,K}K[W_s]$, where reduction $\bmod r$ means here reduction via the R-order $R[W_s]$ in $K[W_s]$.

On $S^\lambda_{\tilde{R}}$ a bilinear form is defined, which has certain invariance properties under the action of $\tilde{R}[W_s]$. In particular, if $\tilde{R} \in \{\bar{R}, K\}$, the radical $S^{\lambda\perp}$ of this bilinear form is a submodule of S^λ, and $D^\lambda = S^\lambda/S^{\lambda\perp}$ is either zero or an irreducible $\tilde{R}[W_s]$-module.

For $\Lambda \in \mathscr{F}$ let e_Λ be the minimal positive integer k such that k divides $1 + q^{\deg\Lambda} + q^{2\deg\Lambda} + \cdots + q^{(k-1)\deg\Lambda}$ and define e to be the cartesian product $\prod e_\Lambda$. If $\lambda_\Lambda = (1^{r_1}, 2^{r_2}, \ldots, k^{r_k})$ is a partition of $k = m_\Lambda(s)$, we say, λ_Λ is e_Λ-regular if either $m_\Lambda(s) = 0$ or $r_i < e_\Lambda$ for $1 \leqslant i \leqslant m_\Lambda(s)$. Call $\lambda = \prod\lambda_\Lambda \in M_s$ e-regular if λ_Λ is e_Λ-regular for all $\Lambda \in \mathscr{F}$. We denote the set of e-regular $\lambda \in M_s$ by $M_{s,e'}$ and $M_s \setminus M_{s,e'}$ by $M_{s,e}$. Note that for all $\Lambda \in \mathscr{F}$, $2 \leqslant e_\Lambda \leqslant r$ and $e_\Lambda = r$ if and only if r divides $q^{\deg\Lambda} - 1$.

On the set of partitions of a natural number define the dominance order $\trianglerighteq$ as in [8, 3.2] and extend this to M_s setting $\lambda \trianglerighteq \mu$ if $\lambda_\Lambda \trianglerighteq \mu_\Lambda$ for all $\Lambda \in \mathscr{F}$, where $\lambda = \prod\lambda_\Lambda$, $\mu = \prod\mu_\Lambda$. Finally fix a linear order $\geqslant$ on M_s compatible with $\trianglerighteq$ (e.g. the lexicographical ordering).

1.2 THEOREM [3, 7.7]. *Let s be a semisimple element of G.*

(i) $S^{\lambda\perp}_K = (0)$ *for all* $\lambda \in M_s$ *and* $\{S^\lambda_K | \lambda \in M_s\}$ *is a full set of nonisomorphic irreducible* $K[W_s]$*-modules.*

(ii) *Let* $\tilde{R} = \bar{R}$, $\lambda \in M_s$. *Then* $D^\lambda \neq (0)$ *if and only if* λ *is e-regular.* $\{D^\mu | \mu \in M_{s,e'}\}$ *is a full set of nonisomorphic irreducible* $\bar{R}[W_s]$*-modules.*

(iii) S^λ_R *is an* $R[W_s]$*-lattice in* S^λ_K *and* $S^\lambda_{\bar{R}} = \bar{R} \otimes_R S^\lambda_R$. *Ordering* $\{S^\lambda_K | \lambda \in M_s\}$ *and* $\{D^\mu | \mu \in M_{s,e'}\}$ *according to the linear order* $\geqslant$ *on* M_s *(downwards), the decomposition matrix* D *of* $R[W_s]$ *is lower unitriangular, i.e.* D *has the form*

$$D = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & * & & 1 \end{pmatrix}.$$

In the following S^λ always means S^λ_R and S^λ_K, $S^\lambda_{\bar{R}}$ is denoted by KS^λ, $\bar{R}S^\lambda$ respectively.

Let $s \in G$ be semisimple satisfying 1.1. Define the subalgebra $A = A_s$ of RG as in [2], and recall that $I \mapsto I(S \otimes 1)\tilde{R}A$ defines an isomorphism between the lattice of all $\tilde{R}$-free right ideals I of $\tilde{R}[W_s]$ and the lattice of all $\tilde{R}S$-homogeneous submodules of $(S \otimes 1)\tilde{R}A = \tilde{R}[W_s](S \otimes 1) = \tilde{R}S^G\varepsilon_s$, where $\tilde{R} \in \{\bar{R}, R, K\}$, by [2, 3.8].

For $\lambda \in M_s$ define the A-module S_λ setting

$$S_\lambda = S^\lambda(S \otimes 1)A = z_\lambda R[W_s](S \otimes 1) \leqslant S^G\varepsilon_s.$$

Then $S_\lambda RG = z_\lambda S^G$ is an RG-submodule of S^G. Moreover it is an RG-lattice in the simple KG-module $z_\lambda KS^G$, which affords the character $\chi_{s,\lambda}$. Set $D^\lambda = \overline{R}S^\lambda/J(\overline{R}S^\lambda)$, $D_\lambda = \overline{R}S_\lambda/J(\overline{R}S_\lambda)$ and $E_\lambda = \overline{R}(z_\lambda S^G)/J(\overline{R}(z_\lambda S^G))$, where, for an arbitrary module M, $J(M)$ denotes the Jacobson radical of M. Now 1.2 and [**2**, 4.6] imply immediately

1.3 Lemma. *Suppose that $s \in G$ satisfies* 1.1, *and let $\lambda \in M_s$ be e-regular. Then D_λ is a simple $\overline{R}S$-homogeneous $\overline{R}A$-module and E_λ is a simple $\overline{R}G$-module. Moreover D_λ occurs in $E_\lambda \varepsilon_s$ as a composition factor.*

Assume 1.1 and let $b_{\lambda\rho}$ be the multiplicity of E_ρ as a composition factor of $\overline{R}(z_\lambda S^G)$, and $a_{\lambda\rho}$ the multiplicity of D^ρ as a composition factor of $\overline{R}S^\lambda$ ($\lambda \in M_s$, $\rho \in M_{s,e'}$). Note that the standard basis theorem [**3**, 5.6 and **2**, 3.7] imply immediately that $\overline{R}(z_\lambda S^G\varepsilon_s) = \bar{z}_\lambda(\overline{R}S^G\varepsilon_s)$ as A-modules, where $\bar{z}_\lambda = z_{\lambda,\overline{R}}$. So we conclude as in [**2**, 5.5/5.6/5.7/5.8] using 1.2:

1.4 Lemma. *Assume* 1.1. *Let $\lambda \in M_s$ and $\rho \in M_{s,e'}$.*

(i) *If $\rho' \in M_{s,e'}$, then $E_\rho \cong E_{\rho'}$ implies $\rho = \rho'$.*

(ii) *$b_{\lambda\rho} \leqslant a_{\lambda\rho}$. In particular $b_{\lambda\rho} \neq 0$ forces $\rho \trianglerighteq \lambda$, and $b_{\rho\rho} = 1$.*

(iii) *Let $Q^\rho \leqslant \overline{R}[W_s]$ be a projective indecomposable $\overline{R}[W_s]$-module such that $Q^\rho/J(Q^\rho) = D^\rho$. Then $J(Q^\rho)\overline{R}S^G$ is the unique maximal submodule of the $\overline{R}G$-module $P_\rho = Q^\rho\overline{R}S^G$. In particular $\tilde{E}_\rho = P_\rho/J(P_\rho)$ is simple and $\tilde{E}_\rho\varepsilon_s = D_\rho$ as $\overline{R}A$-modules.*

For an arbitrary group H with subgroups H_1, H_2 we write $H_1 \leqslant_H H_2$ if $H_1^h \leqslant H_2$ for some $h \in H$.

It is easy to generalize [**2**, 6.1] to the following

1.5 Lemma. *Let s and t be semisimple elements of G and assume that both satisfy* 1.1. *Let $\rho \in M_{s,e'}$ and $\lambda \in M_t$.*

(i) *If E_ρ occurs as a composition factor in $\overline{R}(z_\lambda S_t^G)$ (in $\overline{R}S_t^G$), then $L_t \leqslant_G L_s$.*

(ii) *If $E_\rho\varepsilon_t \neq (0)$, then $L_s \leqslant_G L_t$.*

Remark. Since KS_s affords a cuspidal character of L_s for a semisimple $s \in G$, 1.5 follows easily from [**11**, 4.5] as well.

2. Combinatorics. For $\Lambda \in \mathscr{F}$, let $\tilde{e}_\Lambda$ be the multiplicative order of $q^{\deg \Lambda}$ modulo r. Thus $\tilde{e}_\Lambda = e_\Lambda$ if r does not divide $q^{\deg \Lambda} - 1$, and $\tilde{e}_\Lambda = 1$, $e_\Lambda = r$ otherwise. In particular $1 \leqslant \tilde{e}_\Lambda \leqslant r - 1$. Let ν_r be the exponential valuation of $\mathbf{Z}$ associated to r, normalized so that $\nu_r(r) = 1$. Let $a_\Lambda = \nu_r(q^{\tilde{e}_\Lambda \deg \Lambda} - 1)$ if r is odd. If $r = 2$, then $\tilde{e}_\Lambda = 1$. In this case we set $a_\Lambda = \nu_2(q^{\deg \Lambda} - 1)$ if $q^{\deg \Lambda} = 1 \bmod(4)$ and $a_\Lambda = \nu_2(q^{\deg \Lambda} + 1) - 1$ otherwise. Note that $a_\Lambda \geqslant 2$ for $r = 2$.

2.1 Lemma. *Let i be a positive integer, $\Lambda \in \mathscr{F}$. Then $\nu_r(q^{i \deg \Lambda} - 1) > 0$ if and only if $\tilde{e}_\Lambda$ divides i. If so, then $\nu_r(q^{i\deg \Lambda} - 1) = a_\Lambda + \nu_r(i)$, or $r = 2$, $q = 3 \bmod(4)$ and i is odd. In this case $\nu_2(q^{i \deg \Lambda} - 1) = 1$.*

Proof. This is immediate from the definitions of $\tilde{e}_\Lambda$ and a_Λ. (For odd r compare [**4**, 3A].) $\square$

For $\Lambda \in \mathscr{F}$ let ω be an element of order r^k ($1 \leqslant k \in \mathbf{N}$) in some extension field of $\mathrm{GF}(q^{\deg \Lambda})$, and let Γ be the minimum polynomial of ω over $\mathrm{GF}(q^{\deg \Lambda})$. Define

$i = i_\Lambda(\omega) \in \mathbf{N}$ as follows: If r is odd or $r = 2$ and $q = 1 \bmod(4)$, let $i = k - a_\Lambda$ for $k \geqslant a$ and $i = 0$ otherwise. If $r = 2$ and $q = 3 \bmod(4)$, let $i = 0$ for $k = 1$, $i = 1$ for $2 \leqslant k \leqslant a_\Lambda$ and $i = k - a_\Lambda$ for $a_\Lambda < k$.

2.2 COROLLARY. *Let $\Lambda \in \mathscr{F}$, ω, Γ, k and $i = i_\Lambda(\omega)$ as above. Then* $\deg \Gamma = \tilde{e}_\Lambda r^i$.

For the rest of this section let s be a fixed semisimple element of G, given in its rational canonical form. Moreover we suppose that s is an r'-element, i.e. its order is prime to r. For an arbitrary finite group H we shall denote the set of r-elements of H by H_r. Let $y \in C_G(s)_r$ and $t = sy$. Then t and y are semisimple and $C_G(t) \leqslant C_G(s) = \prod \mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda})$. Let $\prod y_\Lambda$, $\prod t_\Lambda$ and $\prod C_G(s)_\Lambda$ be the corresponding decomposition of y, t and $C_G(s)$ respectively, and let $\Lambda \in \mathscr{F}$ be an elementary divisor of s, i.e. $m_\Lambda(s) \neq 0$. Let $\tilde{\Gamma}$ be an elementary divisor of $y \in \mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda})$, ω a fixed root of $\tilde{\Gamma}$ and σ of Λ. Then the order of ω is a power of r, say r^k. Of course $\mathrm{GF}(q^{\deg \Lambda})[\omega] = \mathrm{GF}(q^{\deg \Lambda \cdot \deg \tilde{\Gamma}}) = \mathrm{GF}(q)[\sigma\omega]$. Let Γ be a minimum polynomial of $\sigma\omega$ over $F = \mathrm{GF}(q)$, and let $i = i_\Lambda(\omega)$ be defined as above. Then by 2.2 $\deg \Gamma = \deg \Lambda \cdot \tilde{e}_\Lambda \cdot r^i$. Now $\mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda}) \leqslant \mathrm{GL}_{m_\Lambda(s)\deg \Lambda}(q) \leqslant G$ canonically, and we may consider t_Λ as an element of $\mathrm{GL}_{m_\Lambda(s)\deg \Lambda}(q)$. We conclude that $m_\Gamma(t_\Lambda)$ equals the multiplicity of $\tilde{\Gamma}$ as an elementary divisor of $y_\Lambda \in \mathrm{GL}_{m_\Lambda(s)}(q^{\deg \Lambda})$, and we get each elementary divisor of t_Λ in this way. In particular, for arbitrary $\Lambda \neq \Gamma \in \mathscr{F}$, $m_\Gamma(t_\Lambda) \neq 0$ implies that $\deg \Gamma = \deg \Lambda \cdot \tilde{e}_\Lambda \cdot r^i$ for some $0 \leqslant i \in \mathbf{Z}$.

Set $N_i = \{\Gamma \in \mathscr{F} \mid \deg \Gamma = \deg \Lambda \cdot \tilde{e}_\Lambda \cdot r^i\}$ and $d_i = \sum_{\Gamma \in N_i} m_\Gamma(t_\Lambda)$. Note that $\Lambda \in N_0$ if and only if $\tilde{e}_\Lambda = 1$. So define d_{-1} to be zero and $N_{-1} = \varnothing$ if $\tilde{e}_\Lambda = 1$, and $d_{-1} = m_\Lambda(t_\Lambda)$ and $N_{-1} = \{\Lambda\}$ otherwise. Then $m_\Lambda(s) = d_{-1} + \tilde{e}_\Lambda \sum_{i=0}^\infty d_i r^i$.

We call t_Λ reduction stable (with respect to r), if there exists at most one $\Gamma \in N_i$ with $m_\Gamma(t_\Lambda) \neq 0$ for all $-1 \leqslant i \in \mathbf{Z}$. Call $t = sy \in sC_G(s)_r$ reduction stable (with respect to r), if t_Λ is reduction stable for all elementary divisors Λ of s, and extend this definition to all semisimple elements t of G in the obvious way. We shall see later that t satisfies 1.1(i) always and 1.1(ii) if and only if t is reduction stable. Note in particular that semisimple r'-elements of G are reduction stable with respect to r.

Let $t = sy$ and Λ be as above. Define D_Λ to be the set of all sequences $(d_i)_{-1 \leqslant i \in \mathbf{Z}}$ of nonnegative integers satisfying $d_{-1} + \tilde{e}_\Lambda \sum_{i=0}^\infty d_i r^i = m_\Lambda(s)$, where $d_{-1} = 0$, if $\tilde{e}_\Lambda = 1$. With $d_\Lambda = (d_i) \in D_\Lambda$ we may associate the partition

$$\lambda_{d_\Lambda} = \left(\deg \Lambda^{d_{-1}}, (\tilde{e}_\Lambda \deg \Lambda)^{d_0}, (\tilde{e}_\Lambda r \deg \Lambda)^{d_1}, \ldots, (\tilde{e}_\Lambda r^k \deg \Lambda)^{d_k}\right)$$

of $k = m_\Lambda(s) \deg \Lambda$. We defined in [**2**, §6] a partial order $\prec$ on the set of partitions of k, which reflects the ordering $\prec$ on the set of Levi subgroups of $\mathrm{GL}_k(q)$ given by $L_1 \prec L_2$ if $L_1 \leqslant_{\mathrm{GL}_k(q)} L_2$, ($L_1$, L_2 are Levi subgroups of $\mathrm{GL}_k(q)$). So define as in [**2**] $d_\Lambda \prec d'_\Lambda$ ($d_\Lambda, d'_\Lambda \in D_\Lambda$) if $\lambda_{d_\Lambda} \prec \lambda_{d'_\Lambda}$.

Let $D_s = \prod D_\Lambda$ (cartesian product) and define a partial order $\prec$ on D_s setting $\mathbf{d} \prec \mathbf{d}'$ ($\mathbf{d} = \prod d_\Lambda$, $\mathbf{d}' = \prod d'_\Lambda$, $d_\Lambda, d'_\Lambda \in D_\Lambda$) if $d_\Lambda \prec d'_\Lambda$ for all $\Lambda \in \mathscr{F}$. Fix a linear order $\leqslant$ on D_s compatible with $\prec$. However $|D_s| = k < \infty$. So we may write the elements of D_s as the increasing chain $\mathbf{d}_1 \lneqq \mathbf{d}_2 \lneqq \cdots \lneqq \mathbf{d}_k$.

So far we have constructed a map from $sC_G(s)_r$ into D_s. For $t = sy$, $y \in C_G(s)_r$, the image of t under this map is denoted by $\mathbf{d}_t$. If $\mathbf{d}_t = \mathbf{d}_j$ ($1 \leqslant j \leqslant k$), we call $j = \mathrm{ht}_s(y)$ the s-height of y. Note that $t' = sy'$ ($y' \in C_G(s)_r$) is conjugate in G to t if and only if y and y' are conjugate in $C_G(s)$. In particular, if t and t' are conjugate in G, then $\mathbf{d}_t = \mathbf{d}_{t'}$. From the definition of $\mathbf{d}_t$ and L_t we get immediately

2.3 LEMMA. *Assume that s has only one elementary divisor, and let $t = sy$, $t' = sy'$ ($y, y' \in C_G(s)_r$). Then $L_t \leqslant_G L_{t'}$ if and only if $\mathbf{d}_t \prec \mathbf{d}_{t'}$. In particular $L_t = L_{t'}$ if and only if* $\mathrm{ht}_s(y) = \mathrm{ht}_s(y')$.

Let s again be an arbitary semisimple r'-element of G, and let $\Lambda \in \mathscr{F}$ be an elementary divisor of s, σ a root of Λ. For $0 \leqslant i \in \mathbf{Z}$ set $j_i = \tilde{e}_\Lambda \deg \Lambda \cdot r^i$. For $2 \leqslant i \in \mathbf{Z}$ fix an element ω_i of order $r^{a_\Lambda + i}$ in $\mathrm{GF}(q^{j_i})$. If r is odd or $r = 2$ and $q = 1$ mod(4), then fix ω_1 in $\mathrm{GF}(q^{j_1})$ of order r^{a_1+1}, and if $r = 2$ and $q = 3$ mod(4), then let ω_1 be an arbitrary 2-element in $\mathrm{GF}(q^{2\deg\Lambda})$ of order $\geqslant 4$. If $\tilde{e}_\Lambda > 1$ fix an element ω_0 of order r^{a_Λ} in $\mathrm{GF}(q^{j_0})$ and set $\omega_{-1} = \sigma$. If $\tilde{e}_\Lambda = 1$, set $\omega_0 = \sigma$ and $\omega_{-1} = 0$. Then $\mathrm{GF}(q^{\deg\Lambda})[\omega_i] = F[\sigma\omega_i] = \mathrm{GF}(q^{j_i})$ for all $0 \leqslant i \in \mathbf{Z}$ with $\omega_i \neq \sigma$. For $0 \leqslant i \in \mathbf{Z}$ with $\omega_i \neq \sigma$ let Λ_i be the minimum polynomial of $\sigma\omega_i$ over F and Λ'_i the minimum polynomial of ω_i over $\mathrm{GF}(q^{\deg\Lambda})$. Set $\Lambda_{-1} = \Lambda$ and $\Lambda'_{-1} = X - 1$ if $\tilde{e}_\Lambda > 1$, and $\Lambda_{-1} = \Lambda'_{-1} = 0$, $\Lambda_0 = \Lambda$ and $\Lambda'_0 = X - 1$ if $e = 1$, where we consider 0 as an irreducible polynomial not occurring as an elementary divisor, i.e. $m_0(t) = 0$ for all semisimple elements t of G.

For $d_\Lambda = (d_i) \in D_\Lambda$ define $y_\Lambda \in C_G(s)_\Lambda$ setting $y_\Lambda = \prod_{i=-1}^{\infty} d_i(\Lambda'_i)$ (matrix direct sum), and for $\mathbf{d} = \prod d_\Lambda \in D_s$ set $y_{\mathbf{d}} = \prod y_\Lambda$. Note that $y_{\mathbf{d}} \in C_G(s)_r$. All elementary divisors of $t_{\mathbf{d}} = sy_{\mathbf{d}}$ are of the form Λ_i for some elementary divisor Λ of s and some $-1 \leqslant i \in \mathbf{Z}$. Moreover $m_{\Lambda_i}(t_{\mathbf{d}}) = d_i$, where $\mathbf{d} = \prod d_\Lambda$ and $d_\Lambda = (d_i) \in D_\Lambda$. Now, for $1 \leqslant j \leqslant k$, let $y_j = y_{\mathbf{d}_j}$ and $t_j = sy_j$. Note in particular that $t_1 = s$, since $\mathbf{d}_1 = \prod d_\Lambda$ with $d_\Lambda = (d_i)$, where $d_{-1} = m_\Lambda(s)$ and $d_i = 0$ $(0 \leqslant i \in \mathbf{Z})$ if $\tilde{e}_\Lambda > 1$, and $d_0 = m_\Lambda(s)$ and $d_{-1} = 0 = d_i$ $(1 \leqslant i \in \mathbf{Z})$ if $\tilde{e}_\Lambda = 1$.

Obviously [2, 6.3] now holds analogously replacing $s\delta(B_s)$ by $sC_G(s)_r$. The following lemma is immediate from the definitions as well:

2.4 LEMMA. *Let s be as above and $1 \leqslant j \leqslant k$. Then t_j is reduction stable.*

For the rest of this paper we shall replace all occurring indices of the form t_j $(1 \leqslant j \leqslant k)$ by j itself, e.g. $W_j = W_{t_j}$, $S_j = S_{t_j}$, etc. Note that in particular $M_1 = M_s$, $M_{1,e'} = M_{s,e'}$ and $W_1 = W_s$.

Let $\lambda = \prod\lambda_\Lambda \in M_1$, i.e. $\lambda_\Lambda \vdash m_\Lambda(s)$ $(\lambda \in \mathscr{F})$. Let $\lambda_\Lambda = (1^{k_1}, 2^{k_2}, \ldots, m^{k_m})$, where $m = m_\Lambda(s)$. Then $\sum_{\nu=1}^{m} k_\nu \nu = m$. Let $k_\nu = \tilde{e}_\Lambda \tilde{k}_\nu + d_{-1}$ for some $d_{-1} \in \mathbf{Z}$ with $0 \leqslant d_{-1} < \tilde{e}_\Lambda$, and let $\tilde{k}_\nu = \sum_{i=0}^{\infty} d_i^{(\nu)} r^i$ be the r-adic expansion of k_ν. Set $d_i = \sum_{\nu=1}^{m} d_i^{(\nu)}\nu$. Then $d_{-1} + \tilde{e}_\Lambda \sum_{i=0}^{\infty} d_i r^i = m = m_\Lambda(s)$. Thus $d_\Lambda = (d_i)_{-1 \leqslant i \in \mathbf{Z}} \in D_\Lambda$ and $\mathbf{d} = \prod d_\Lambda \in D_s$, i.e. $\mathbf{d} = \mathbf{d}_j$ for some $1 \leqslant j \leqslant k$. By [2, 6.3] $m_{\Lambda_i}(t_j) = d_i$, and obviously

$$d_i = \sum_{\nu=1}^{m} d_i^{(\nu)}\nu = \sum_{\nu=1}^{d_i} d_i^{(\nu)}\nu.$$

Therefore $\lambda_{\Lambda_i} = (1^{d_i^{(1)}}, \ldots, \nu^{d_i^{(\nu)}}, \ldots, k^{d_i^{(k)}})$ is a partition of $d_i = k$. If $\tilde{e}_\Lambda = 1$, then $e_{\Lambda_i} = r$ for $0 \leqslant i \in \mathbf{Z}$, and $d_{-1} = 0$. By construction, $0 \leqslant d_i^{(\nu)} < r$ for all $1 \leqslant \nu \leqslant d_i$, hence λ_{Λ_i} is e_{Λ_i}-regular. If $\tilde{e}_\Lambda > 1$, then $e_{\Lambda_i} = r$ for $0 \leqslant i \in \mathbf{Z}$ by 2.1, and $\tilde{e}_\Lambda = e_\Lambda = e_{\Lambda_{-1}} < r$. By construction $d_{-1} < \tilde{e}_\Lambda = e_{\Lambda_{-1}}$ and $0 \leqslant d_i^{(\nu)} < e_{\Lambda_i} = r$ for all $1 \leqslant \nu \leqslant d_i$, hence in all cases λ_{Λ_i} is e_{Λ_i}-regular for all $-1 \leqslant i \in \mathbf{Z}$. Taking the cartesian product over all $\Lambda \in \mathscr{F}$ and all $-1 \leqslant i \in \mathbf{Z}$ we have defined an e-regular element μ_λ of M_j, i.e. $\mu_\lambda \in M_{j,e'}$. This defines a map $\lambda \mapsto \mu_\lambda$ from M_1 into $\bigcup_{j=1}^{k} M_{j,e'}$. Note that this union is disjoint by [2, 6.3].

As in [2] an inverse mapping may be constructed easily, so [2, 6.4] holds analogously, i.e. the map constructed above is a bijection. In the following we will identify M_1 with $\bigcup_{j=1}^{k} M_{j,e'}$ by this bijection.

3. The theorem. Fix a semisimple r'-element s of G. Let B_s be the union of all blocks of RG with semisimple part s, where the semisimple part of an r-block of G is defined as in [4, 5D]. It should be remarked that $\tilde{e} = \prod \tilde{e}_\Lambda$ plays an important role in the classification of the r-blocks of G, where $\tilde{e}_\Lambda$ ($\Lambda \in \mathscr{F}$) is defined as in §2. In fact, the blocks, which are contained in B_s, may be parametrized by $\tilde{e}$-cores of elements of M_s, the so-called unipotent parts of the blocks. Moreover, if $B_{s,\lambda}$ denotes the r-block of G with semisimple part s and unipotent part λ, a defect group $\delta(B_{s,\lambda})$ of $B_{s,\lambda}$, contained in $C_G(s)$, has been determined in [4, 5D]. P. Fong and B. Srinivasan showed in [4, 7A] as well, that $\{\chi_{sy,\mu} | y \in \delta(B_{s,\lambda})$, the e-core of $\mu = \lambda\}$ is the set of irreducible characters in $B_{s,\lambda}$, and in [4, 8A] that the set of characters of the form $\chi_{s,\mu}$ ($\mu \in M_s$) in $B_{s,\lambda}$ restricted to $G_{r'}$ span the $\mathbf{Z}$-module generated by the irreducible Brauer characters in $B_{s,\lambda}$. They proved this all under the additional condition that r is odd. Recently M. Broue and L. Puig found an entirely different proof of it, which works for the prime two as well [1]. But the prime two case can be worked out too, using P. Fong's and B. Srinivasan's methods, replacing in several places arguments which work only for odd primes r by ad hoc arguments for $r = 2$ (e.g. in [4, 3D]).

Let us return to the union B_s of blocks with semisimple part s. By [2, 2.1/2.3] s satisfies 1.1. It is easy to extend [2, 6.7] to the following:

3.1 LEMMA. *Let $y \in C_G(s)_r$, $t = sy$, and assume that t is reduction stable. Let $y' \in C_G(s)_r \cap L_t$ and $t' = sy'$. Then t and t' are conjugate in G if and only if they are conjugate in L_t.*

Let $t = sy$, $y \in C_G(s)_r$. Let $V \in b_t$ be an irreducible constituent of $KS_{tP_t}^{G}$ and ψ the character afforded by V. Denote the restriction of ψ to L_t by χ. Since the semisimple part of b_t is s (where we identify blocks of L_t with their liftings to P_t), $\chi \in (sy')^{L_t}$ for some $y' \in C_{L_t}(s)_r$. Thus the KL_t-module affording χ is a constituent of the KL_t-module $KS_{sy'}^{L_t}$, replacing G by L_t.

Using [2, 1.2], the orthogonality relations for Deligne-Lusztig operators [11, 6.14] and Frobenius reciprocity we see that sy' and t has to be conjugate in G (compare the proof of [2, 2.3] and [2, 6.9]). If t is reduction stable, t and sy' has to be conjugate in L_t by 3.1, hence $L_t = L_{sy'}$ by construction of L_t and $L_{sy'}$ respectively. In particular $KS_{sy'}^{L_t} = V_{L_t}$ is irreducible and $V \not\cong KS_t$, showing that t satisfies 1.1(ii).

Assume that t is not reduction stable. So let $\Lambda \in \mathscr{F}$, $m_\Lambda(s) \neq 0$ and $-1 \leqslant i \in \mathbf{Z}$ such that there exist two different elementary divisors of t in N_i, say Γ, Γ'. Then $t = B + (\Gamma) + (\Gamma') + C$ (matrix direct sum) for some matrices B and C. Let $t' \in G$ be defined setting $t' = B + (\Gamma') + (\Gamma) + C$. Then t and t' are conjugate in G but not in L_t. Moreover by [4, 8A] S_t and $S_{t'}$ are both contained in the block b_t of RP_t.

Using [**2**, 1.2, **11**, 6.14] and Frobenius reciprocity once more, we see that the irreducible KP_t-module $KS_{t'}$ contained in b_t is different from KS_t and occurs as a constituent of $KS_{tP_t}^G$. We have shown

3.2 LEMMA. *Let $t \in G$ be semisimple. Then t satisfies* 1.1(ii) *if and only if t is reduction stable with respect to r.*

3.3 LEMMA. *Let $t \in G$ be semisimple. Then t satisfies* 1.1(i). *Moreover $\overline{R}S_t \cong \overline{R}S_j$, where $j = \mathrm{ht}_s(y)$ for the r'-part s and r-part y of t.*

PROOF. We have to consider S_t as an RL_t-module contained in the block b_t of L_t. So we may assume that t has only one elementary divisor Γ of degree n. Consequently s has only one elementary divisor as well, say Λ. If $\deg \Gamma = \deg \Lambda$, then $L_s = L_t = G$ and B_s contains just the block b_t. By [**4**, 7A] b_t has a cyclic defect group and contains only one irreducible Brauer character by [**4**, 8A]. From this the assertion follows immediately.

So assume that $\deg \Gamma > \deg \Lambda$. Then $n = \tilde{e}_\Lambda \deg \Lambda \cdot r^h$, $s = \tilde{e}_\Lambda r^h(\Lambda)$ and $t = (\Gamma)$ for some $0 \leqslant h \in \mathbf{Z}$. Let $t' = (\Lambda_h) \in G$, where $\Lambda_h \in \mathscr{F}$ is defined as in §2. Then $L_t = L_{t'} = G$, $\mathbf{d}_t = \mathbf{d}_{t'}$ and $\mathrm{ht}_s(y) = \mathrm{ht}_s(y')$, where y' denotes the r-part of t'. Moreover $M_t = M_{t,e'}$ contains just one element, say $\lambda = \prod \lambda_{\Gamma'}$, with $\lambda_\Gamma = (1)$ and $\lambda_{\Gamma'} = (-)$ for $\Gamma \neq \Gamma' \in \mathscr{F}$.

As in §2 let $D_s = \{\mathbf{d}_1, \ldots, \mathbf{d}_k\}$. Then $\mathrm{ht}_s(y) = k$. We may assume by induction on n that the assertion holds for all L_j, $1 \leqslant j \leqslant k-1$, since $L_j = L_k$ implies $j = k$ by 2.3. By 2.4 and 3.2 t_j $(1 \leqslant j \leqslant k-1)$ satisfies 1.1. So we have defined by 1.3 for each $\mu \in M_s = M_1 = \bigcup_{j=1}^k M_{j,e'}$ with $\mu \neq \lambda$ an irreducible $\overline{R}G$-module E_μ contained in B_s. Let $\lambda \neq \mu$, $\mu' \in M_1$ and assume that $E_\mu \cong E_{\mu'}$. Let $\mu \in M_{j,e'}$, $\mu' \in M_{j',e'}$, $1 \leqslant j, j' \leqslant k-1$. Then by 1.5 $L_j \leqslant_G L_{j'}$ and $L_{j'} \leqslant_G L_j$, hence $j = j'$ by 2.3, and then $\mu = \mu'$ by 1.4. So far we have found all but one irreducible $\overline{R}G$-module in B_s by [**4**, 8A]. Let E be the remaining irreducible $\overline{R}G$-module in B_s. By 1.5 E_μ does not occur as a composition factor of $\overline{R}S_t = \overline{R}S_t^G$ for $\lambda \neq \mu \in M_1$. Thus E is the only composition factor of $\overline{R}S_t$, and as in [**2**, 6.9] we conclude from [**2**, 6.5/6.6] that the multiplicity of E as composition factor of $\overline{R}S_t$ is one, showing that $\overline{R}S_t = E$ is irreducible. This applies to t' as well, so $\overline{R}S_t = \overline{R}S_{t'} = E$, and the lemma is proved. □

As an immediate consequence we get the following theorems:

3.4 THEOREM. *Let $t \in G$ be semisimple. Then t satisfies* 1.1 *if and only if t is reduction stable. If the r-part of t has s-height j (s the r'-part of t), then* $\mathrm{End}_{\overline{R}G}(\overline{R}S_t^G) = \overline{R}[W_j]$. *Finally* $\overline{R} \otimes_R \mathrm{End}_{RG}(S_t^G) = \mathrm{End}_{\overline{R}G}(\overline{R}S_t^G)$ *if and only if t is reduction stable.*

3.5 THEOREM. *Let χ be an irreducible cuspidal character of G. Then χ restricted to $G_{r'}$ is an irreducible Brauer character for all primes r not dividing q ($G = \mathrm{GL}_n(q)$).*

3.6 COROLLARY. *Let s be a semisimple r'-element of G, and let t_j $(1 \leqslant j \leqslant k)$ be defined as in* §2. *Then t_j satisfies* 1.1 *for all $j \in \{1, \ldots, k\}$.*

So far we have defined for each $\rho \in M_1$ an irreducible $\overline{R}G$-module E_ρ by 3.6, 1.3 and [**2**, 6.4]. We proceed now as in [**2**] to prove our main theorem: First we use 1.3, 2.3, 1.5 and 1.4(i) to show that $\{E_\rho | \rho \in M_1\}$ is a full set of nonisomorphic irreducible $\overline{R}G$-modules in B_s, provided s has only one elementary divisor [2, 6.10]. Then $E_\rho \cong \tilde{E}_\rho$ as well, where $\tilde{E}_\rho$ ($\rho \in M_1$) is defined as in 1.4(iii) [2, 6.11].

Next we denote for an arbitrary semisimple r'-element s of G the multiplicity of E_ρ as a composition factor of $\overline{R}(z_\lambda S_{j'}^G)$ by $d_{\lambda\rho}$, where $\rho \in M_{j,e'}$, $\lambda \in M_{j'}$, $1 \leqslant j, j' \leqslant k$.

The $(|M_{j'}| \times |M_{j,e'}|)$-matrix $(d_{\lambda\rho})$ is denoted by $D_{j'j}$ and the decomposition matrix of $R[W_j]$ by D^j. Replacing r' by e' the proofs of [2, 6.12 and 2, 6.13] carry over word by word, showing the following theorems.

3.7 **Theorem.** *Let s be a semisimple element of G. Then the following holds*:

(i) $\{E_\rho | \rho \in M_1\}$ *is a full set of nonisomorphic irreducible $\overline{R}G$-modules in the union B_s of r-blocks of G with semisimple part s.*

(ii) *Let* $1 \leqslant j, j' \leqslant k$, $\rho \in M_{j,e'}$, $\lambda \in M_j$. *Then* $d_{\lambda\rho} \neq 0$ *implies that* $\mathbf{d}_{j'} \prec \mathbf{d}_j$. *In particular, if $j < j'$, then $D_{j'j} = 0$.*

(iii) $D_{jj} = D^j$ *for all* $j \in \{1,\ldots,k\}$.

3.8 **Theorem.** *Let s be a semisimple r'-element of G and let B_s be the union of all r-blocks of G with semisimple part s. Then there exist* $1 = y_1, y_2,\ldots,y_k \in C_G(s)_r$ *and a bijection between M_s and the disjoint union* $\bigcup_{j=1}^k M_{j,e'}$ *such that* $\{E_\rho | \rho \in M_{j,e'}, 1 \leqslant j \leqslant k\}$ *is a full set of nonisomorphic irreducible $\overline{R}G$-modules in B_s. Ordering $1,\ldots,k$ downwards, the decomposition matrix D of B_s has the following form*:

$$D = \begin{pmatrix} D^k & & 0 & \\ & D^{k-1} & & \\ & & \ddots & D^1 \\ & * & & \end{pmatrix},$$

where D^j $(1 \leqslant j \leqslant k)$ *denotes the decomposition matrix of the Hecke algebra* $R[W_j]$.

Furthermore, the irreducible characters and Brauer characters in B_s may be ordered so that D is lower unitriangular, i.e. has the form

$$D = \begin{pmatrix} 1 & & 0 & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & * & & \end{pmatrix}.$$

3.9 **Corollary.** *Let G be a finite general linear group defined over a field F of characteristic p. Let $r \neq p$ be a prime. Then the decomposition matrix of G with respect to r is lower unitriangular for a suitable ordering of the irreducible characters and Brauer characters of G.*

However, [2, 6.13] generalizes as well:

3.10 **Corollary.** *Let* $s = \prod m_\Lambda(s)(\Lambda)$ *be a semisimple r'-element of* $G = \mathrm{GL}_n(q)$. *Define the Levi subgroup $\tilde{L}$ of G setting* $\tilde{L} = \prod \mathrm{GL}_{m_\Lambda(s)\deg\Lambda}(q)$ *and let b be the union* *of all blocks of $\tilde{L}$ with semisimple part s. Let B_s be defined as in* 3.8. *Then the following hold*:

(i) *Let V_0 be an $\overline{R}\tilde{L}$-module in b and $V \leqslant V_0$. Then $V \mapsto V^G$ defines an isomorphism from the submodule lattice of V_0 onto the submodule lattice of V_0^G.*

(ii) *The decomposition matrices of b and B_s coincide.*

Let s be as in 3.10, $y \in C_G(s)_r$ and $t = sy$. Let $j = \mathrm{ht}_s(y)$. By [**4**, 8A] $(t)^G \subseteq B_s$. So we may ask what we can say about the part of the decomposition matrix D of B_s which corresponds to the geometric conjugacy class $(t)^G$. We have seen in 3.3 that $\overline{R}S_t \cong \overline{R}S_j$. Let $\lambda \in M_t$. By [**2**, 4.5] $z_\lambda KS_t^G$ affords the irreducible character $\chi_{t,\lambda}$ of G. As in the first section let b_t be the block of L_t containing KS_t, and ε_t be the block idempotent of b_t. However $L_t = L_j$ by construction of $\mathbf{d}_t$, L_t and ht_s, and $\varepsilon_t = \varepsilon_j$, since $\overline{R}S_t \cong \overline{R}S_j$.

Now $\overline{R}(z_\lambda S_t^G)$ is a reduction modulo r of $z_\lambda KS_t^G$. Moreover we have seen in the first section that $\overline{R}(z_\lambda S_t^G \varepsilon_t) = \bar{z}_\lambda(\overline{R}S_t^G \varepsilon_t)$ as $\overline{R}A_t$-modules, where the subalgebra A_t of RG is defined as in [**2**, §3].

Since $\varepsilon_t = \varepsilon_j$, $\bar{z}_\lambda(\overline{R}S_t^G\varepsilon_t)$ is an $\overline{R}A_j$-submodule of $\overline{R}S_t^G$ as well. By 3.10 we may assume that s has only one elementary divisor. So we conclude as in [**2**, 6.12] that the multiplicity $d_{\lambda\rho}$ of E_ρ $(\rho \in M_{j,e'})$ as a composition factor of $\overline{R}(z_\lambda S_t^G)$ equals the multiplicity of the $\overline{R}A_j$-module D_ρ as a composition factor of $\bar{z}_\lambda(\overline{R}S_t^G\varepsilon_t)$.

By [**2**, 3.7], the latter equals the multiplicity of the $\overline{R}[W_j]$-module D^ρ as a composition factor of $\bar{z}_\lambda \overline{R}[W_j]$. However $\overline{R}[W_t]$ is a standard Young subalgebra of $\overline{R}[W_j]$. Now [**3**, 7.4] implies immediately that $\bar{z}_\lambda \overline{R}[W_j] \cong \overline{R}S^\lambda \otimes_{\overline{R}[W_t]} \overline{R}[W_t]$ has a Specht series (independent of q), which may be calculated by the Littlewood-Richardson rule [**8**, 16.4].

Denote the $(|M_t| \times |M_{j,e'}|)$-matrix $(d_{\lambda\rho})$ by D_{tj} and let $d^{\lambda\mu}$ be the multiplicity of $\overline{R}S^\mu$ in a Specht series of $\bar{z}_\lambda \overline{R}[W_j]$ $(\mu \in M_j)$.

3.11 THEOREM. *Let s be a semisimple r'-element of G, $y \in C_G(s)_r$ and $t = sy$. Let $j = \mathrm{ht}_s(y)$. Then the following hold*:

(i) *Let $\rho \in M_{j',e'}$, $1 \leqslant j' \leqslant k$, $\lambda \in M_t$ and assume that E_ρ occurs as a composition factor in $\overline{R}(z_\lambda S_t^G)$. Then $\mathbf{d}_j \prec \mathbf{d}_{j'}$.*

(ii) *Let $\lambda \in M_t$, and assume that $\overline{R}S^\mu$ occurs $d^{\lambda\mu}$ times in a Specht series of $\bar{z}_\lambda \overline{R}[W_j]$. Then the row of D_{tj} corresponding to $\lambda \in M_t$ is $\sum_{\mu \in M_j} d^{\lambda\mu} r_\mu$, where r_μ is the row of $D_{jj} = D^j$ corresponding to $\mu \in M_j$.*

PROOF. (i) follows as in [**2**, 6.12] using (3.10), and (ii) has been proved above. □

REMARK. We do not say anything about the multiplicity of E_ρ $(\rho \in M_{j',e'})$ as a composition factor of $\overline{R}(z_\lambda S_t^G)$ for $j' < j$. The result above indicates that this is given by the Littlewood-Richardson rule as well, i.e. it may be conjectured, that 3.11(ii) holds analogously replacing the matrices D_{tj} and D_{jj} by the part of the decomposition matrix of B_s which corresponds to the geometric conjugacy class $(t)^G$, $(t_j)^G$ respectively.

In [**9**] G. James defined for the special case $s = 1$, $\lambda \in M_s$, an analogue to the Specht module S^λ in a characteristic free way for the general linear group. He proved as well an analogue to the Specht series theorem [**8**, Chapter 17] (compare

[**10**, §15]). This implies in particular that the Littlewood-Richardson rule holds in fact for the unipotent of G and for their reductions modulo r. This gives a good reason that the conjecture above is true.

As a final application we get

3.12 THEOREM. *Let P be a parabolic subgroup of G with Levi complement L. Let V be an irreducible RL-module, which affords a cuspidal character of L, lifted to P. Then all indecomposable direct summands of $\overline{R}V^G$ are liftable.*

PROOF. First of all we may assume that $L = L_t$ for some semisimple t of G and $V = S_t$. Since $\overline{R}S_t \cong \overline{R}S_{t'}$ for some reduction stable t' in G by 3.3, we may assume that t is reduction stable. So t satisfies 1.1 by 3.4, in particular $\operatorname{End}_{\overline{R}G}(\overline{R}S_t^G) = \overline{R} \otimes_R \operatorname{End}_{RG}(S_t^G)$. As in the case of permutation modules, this implies immediately the assertion of the theorem. □

REMARK. Obviously a direct indecomposable summand of $\overline{R}S_t^G$ may be lifted to a direct summand of S_t^G provided t is reduction stable. This does not hold, in general, if t is not reduction stable.

The indecomposable direct summands of RS_t^G correspond bijectively to the indecomposable direct summands of the endomorphism ring of RS_t^G, i.e. to the projective indecomposables of $\overline{R}[W_t]$ (t reduction stable). So, if $\rho \in M_{t,e'}$, then the character of $K\tilde{P}_\rho$, where $\overline{R}\tilde{P}_\rho = P_\rho$ and the $\overline{R}G$-module P_ρ is defined as in 1.4(iii), is given by the column of the decomposition matrix of $R[W_t]$ corresponding to $\rho \in M_{t,e'}$.

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